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# Time-dependent response of a floating flexible plate to an impulsively started steadily moving load

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The time-dependent response of a floating flexible plate to an impulsively started steadily moving load defines the time taken to approach a steady-state deflection due to the load, or indeed whether such a steady state is achieved at all. The asymptotic analysis for large time reported here, for both a concentrated point load and a uniformly distributed circular load, confirms that a steady-state deflection is achieved at both subcritical and supercritical load speeds. This analysis also predicts a logarithmically growing response near the critical speed corresponding to the minimum phase speed of the hybrid waves generated, but an eventual steady-state response when the load speed moves at the shallow water wave speed. These results are supported by numerical computation.

## 1. Introduction

Theory for the deflection of a continuously supported beam or plate due to a moving load is applicable to various transport systems. The beam or plate may represent a rail or road surface, an airport runway or a floating ice sheet in a cold region. The moving load might be a conventional vehicle, a landing aeroplane or a hovercraft. Moving loads on ice plates is the subject of a recent monograph (Squire *et al.* 1996), where it is emphasized that the deflection can be much greater when the load is moving than when it is stationary. Kerr (1983) pointed out that the minimum phase speed $c_{min}$ of hybrid waves, largely determined by the flexural rigidity of the plate and the properties of the underlying foundation, coincided with a classical critical speed. Davys, Hosking & Sneyd (1985) subsequently noted that wave energy may accumulate beneath a load travelling at or near speeds coincident with both the group speed and the phase speed of generated waves. They also showed *inter alia* that gravity-dominated trailing waves may propagate away behind a localized load moving over a floating ice plate only if $V \cos\beta < (gH)^{1/2}$, where $V$ is the load speed and $\beta$ is the angle that the normal to any wave crest makes with the direction in which the load is moving. The water

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wave speed $(gH)^{1/2}$, where $g$ is the gravitational acceleration and $H$ is the depth of the underlying water, is thus another critical speed candidate. Leading shorter flexural waves correspond to wavenumbers where the group speed exceeds the phase speed, and longer gravity-dominated waves to wavenumbers where the reverse is the case.

In an analysis of the response of a thin floating elastic plate to a steadily moving concentrated line load, on the implicit assumption that a steady state exists, Kheysin (1967) originally identified two critical speeds where the deflection is theoretically infinite. Nevel (1970) subsequently analysed the deflection due to a steadily moving load uniformly distributed over a circular area, and *inter alia* noted that his integral for the deflection directly beneath the load centre is unbounded at a particular load speed, so it emerged that such singularities were not due to load concentration or dimensionality.

Kheysin (1971) recognized that a time-dependent analysis might help explain the singular deflection at critical speed. Thus he set out to examine whether or not the deflection, evidently so dependent on the load speed, actually approaches a steady state – and if it does, to ascertain the time taken for transients to die out. He found that an impulsively started concentrated line load produces a deflection eventually growing in time as $t^{1/2}$ when $V = c_{min}$, consistent with continuous energy accumulation under the load at that speed (Davys *et al.* 1985). Schulkes & Sneyd (1988) provided a much more complete asymptotic analysis for an impulsively started concentrated line load, confirming this $O(t^{1/2})$ growth when $V = c_{min}$ and showing *inter alia* that the deflection also grows as $t^{1/3}$ when $V = (gH)^{1/2}$ (as $t \to \infty$).

Steady wave patterns, originally derived by Davys *et al.* (1985) for a uniformly moving point load, have largely been reproduced for uniform rectangular load distributions by Milinazzo, Shinbrot & Evans (1995), with some small variations depending on load aspect ratio. However, they noted that whereas no bounded steady state is possible when $V = c_{min}$ (in the absence of dissipation), their steady-state solution when $V = (gH)^{1/2}$ appears bounded. Squire *et al.* (1996) nevertheless chose to refer to $(gH)^{1/2}$ as 'critical', pending the development of further relevant time-dependent analysis.

In this paper, we discuss the response of a floating flexible plate to an impulsively started concentrated point load and an impulsively started load uniformly distributed over a circular area, to investigate in particular whether the time-dependent deflection at various load speeds $V$ differs from that found by Schulkes & Sneyd (1988) for the one-dimensional case of a line load. The mathematical model we use is outlined in detail in § 2.

## 2. Mathematical model

We consider a thin elastic homogeneous plate of infinite extent, with thickness $h$ and density $\rho'$ floating on water of uniform depth $H$. The upper undisturbed water surface is at $z = 0$ and the seafloor is at $z = -H$, where the $(x, y)$-plane coincides with the thin plate. The water density is denoted by $\rho$. If $f(x, y, t)$ denotes the moving load on the plate, then the linearized equation for the vertical plate deflection $\eta(x, y, t)$ is (see for example Davys *et al.* 1985; Schulkes & Sneyd 1988)

$$D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\eta+\rho' h\frac{\partial^2\eta}{\partial t^2}=-\rho(\phi_t)_{y=0}-\rho g\eta-f(x,y,t)\quad\text{for} -\infty<x,y<\infty,\quad(2.1)$$

where the gravitational acceleration $g$ is in the negative $y$-direction and $\phi(x, y, z, t)$ represents the velocity potential for irrotational and incompressible flow in the water. The velocity potential thus satisfies the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \quad -H < z < 0, \tag{2.2}$$

subject to the boundary conditions $(\partial\phi/\partial y)(x, y, -H, t) = 0$ and the kinematic (non-cavitation) condition $(\partial\phi/\partial z)(x, y, 0, t) = (\partial\eta/\partial t)|_{z=0}$. The flexural rigidity coefficient for a thin elastic plate is commonly given in terms of Young's modulus $E$ and Poisson's ratio $\nu$ as

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

although corrections can be made, for example to account theoretically for plate inhomogeneity (Kerr & Palmer 1972; Squire *et al.* 1996).

## 3. Time-dependent deflection due to a point load

By considering a line load, Schulkes & Sneyd (1988) reduced the problem to one spatial dimension. In this section, we consider a moving concentrated point load, and then a uniformly distributed circular load in the next section, to discuss the time development of disturbances that may propagate in directions other than in the line of motion of the load.

Taking an impulsively started point load subsequently travelling with uniform speed $V$ in the positive $x$-direction, we write $f(x, y, t) = P_0\delta(x - Vt)\delta(y)U(t)$ where $P_0$ is the load pressure per unit area, $\delta$ is the Dirac delta function and

$$U(t) = \begin{cases} 0, & t \leqslant 0 \\ 1, & t > 0 \end{cases}$$

is the Heaviside unit step function. Then taking the Fourier transform of (2.1) and (2.2) in $x$ and $y$ denoted by the hat superscript, we obtain

$$Dk^4\hat{\eta} + \rho g\hat{\eta} + \rho' h\hat{\eta}_{tt} + \rho\hat{\phi}_t(k, 0, t) = -\frac{P_0}{2\pi}\mathrm{e}^{-\mathrm{i}k_1 Vt} \tag{3.1}$$

for $t > 0$, with

$$\hat{\phi}_{zz} - k^2\hat{\phi} = 0, \quad \hat{\phi}_z(k, -H, t) = 0, \quad \hat{\phi}_z(k, 0, t) = \hat{\eta}_t, \tag{3.2}$$

where $k = (k_1^2 + k_2^2)^{1/2}$ is the wavenumber with Cartesian components $(k_1, k_2)$, and the $t$ and $z$ subscripts indicate partial differentiation. Solving (3.2), substituting the result into (3.1) and using the initial conditions $(\hat{\eta})_{t=0} = 0$, $(\hat{\eta}_t)_{t=0} = 0$, we obtain

$$\left(\rho' h + \frac{\rho}{k}\coth(kH)\right)\hat{\eta}_{tt} + Dk^4\hat{\eta} + \rho g\hat{\eta} = -\frac{P_0}{2\pi}\mathrm{e}^{-\mathrm{i}k_1 Vt}. \tag{3.3}$$

Given that we are interested in wavelengths much larger than the plate thickness $h$, we neglect the plate acceleration term with coefficient $\rho' h$ in (3.3). Upon solving the differential equation (3.3) we thus obtain

$$\hat{\eta}(k, t) = -\frac{P_0}{4\pi\rho}\frac{\mathrm{e}^{-\mathrm{i}k_1 Vt}}{c(k)}\left[\frac{(1 - \mathrm{e}^{-\mathrm{i}\Psi_1 t})}{\Psi_1} + \frac{(1 - \mathrm{e}^{\mathrm{i}\Psi_2 t})}{\Psi_2}\right]\tanh(kH), \tag{3.4}$$

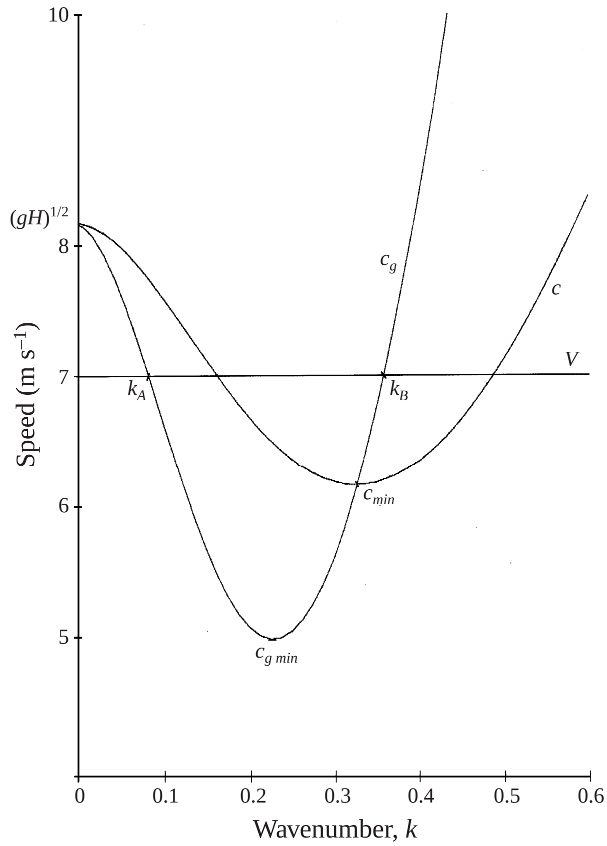


FIGURE 1. Dispersion relation; the parameters are $D = 2.5 \times 10^5$, $\nu = \frac{1}{3}$, $h = 0.175$ m, $H = 6.8$ m, $g = 9.8$ m s$^{-2}$.

where $\Psi_1 = kc - k_1 V$ and $\Psi_2 = kc + k_1 V$ are suitable phase functions. The phase speed $c = \omega/k$, where $\omega$ is the angular frequency, is given by the dispersion relation (see also figure 1)

$$c^2 = \left(\frac{Dk^4}{\rho} + g\right)\frac{\tanh(kH)}{k}. \tag{3.5}$$

It is convenient to introduce the coordinate $X = x - Vt$ corresponding to a reference frame moving with the load, so the expression for the plate deflection obtained by the inverse Fourier transform is

$$\eta(X, y, t) = -\frac{P_0}{8\pi^2\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\tanh(kH)}{c(k)}\mathrm{e}^{\mathrm{i}(k_1X + k_2y)} \times \left[\frac{(1 - \mathrm{e}^{-\mathrm{i}\Psi_1 t})}{\Psi_1} + \frac{(1 - \mathrm{e}^{\mathrm{i}\Psi_2 t})}{\Psi_2}\right]\mathrm{d}k_1\mathrm{d}k_2. \tag{3.6}$$

The phase speed $c(k)$ is positive for real $k$ and the zeros of the denominators $\Psi_1$, $\Psi_2$ are also zeros of the respective numerators, so this expression is analytic in some neighbourhood of the real axis in the $(k_1k_2)$-plane. Some direct numerical computation using (3.6) is described later. To get asymptotic expansions for large time, we need only consider the term involving $\Psi_1$, since the other phase function $\Psi_2$

is monotonically increasing with $k$. We prefer to use an equivalent polar form for this purpose, however.

Using the polar coordinate transformation $X = r\cos\xi$, $y = r\sin\xi$, $k_1 = k\cos\theta$, $k_2 = k\sin\theta$, we re-express (3.6) as

$$\eta(r,\xi,t) = -\frac{P_0}{4\pi^2\rho}\mathrm{Re}\left(\int_0^\infty \int_{-\pi/2}^{\pi/2} \frac{\tanh(kH)}{c(k)} \mathrm{e}^{\mathrm{i}kr\cos(\theta-\xi)} \times \left\{ \frac{1-\mathrm{e}^{-\mathrm{i}k(c-V\cos\theta)t}}{c-V\cos\theta} + \frac{1-\mathrm{e}^{\mathrm{i}k(c+V\cos\theta)t}}{c+V\cos\theta} \right\} \mathrm{d}\theta \mathrm{d}k\right) \tag{3.7}$$

where $\mathrm{Re}(z)$ denotes the real part of $z$. An alternative expression for the deflection in these polar coordinates is

$$\begin{aligned}\eta(r,\xi,t) &= -\frac{P_0}{4\pi^2\rho}\int_0^\infty \int_{-\pi/2}^{\pi/2} \frac{k\tanh(kH)}{c(k)} \int_0^t \{\sin\left[kr\cos\left(\theta-\xi\right)+k(c+V\cos\theta)s\right] \\ &\quad -\sin[kr\cos(\theta-\xi)-k(c-V\cos\theta)s]\}\mathrm{d}s\mathrm{d}\theta\mathrm{d}k \\ &= -\frac{P_0}{2\pi\rho}\int_0^\infty \frac{k\tanh\left(kH\right)}{c(k)} \int_0^t J_0(kA)\sin\left(kcs\right)\mathrm{d}s\mathrm{d}k \end{aligned} \tag{3.8}$$

where $A = [(r\cos\xi + Vs)^2 + r^2\sin^2\xi]^{1/2}$. Both (3.7) and (3.8) define the deflection at any field point $(r,\xi)$ in the plane of the flexible plate. For the ultimate steady state, (3.8) can be used to show that the maximum deflection occurs at the concentrated point load for load speeds $V < c(k)$ (i.e. at $r = 0$ where $A = Vs$), as one might have anticipated in the absence of visco-elasticity (Takizawa 1988; Squire *et al.* 1996).

## 4. Local deflection due to a distributed circular load

Let us now generalize the idea of Nevel (1970), to obtain the time-dependent deflection under the centre of a load uniformly distributed over a circular area of radius $R$. Thus we introduce

$$\eta(t) = \int_0^R \int_0^{2\pi} \frac{\eta(r,\xi,t)}{\pi R^2} r\mathrm{d}r\mathrm{d}\xi,$$

so from (3.8) we have

$$\eta(t) = -\frac{P_0}{\pi\rho R}\int_0^\infty \frac{\tanh(kH)J_1(kR)}{c(k)} \int_0^t J_0(kVs)\sin(kcs)\mathrm{d}s\mathrm{d}k. \tag{4.1}$$

We note that the Bessel function $J_1(kR)$ in this result characterizes the distributed load, in comparison with (3.8) for the concentrated point load.

The eventual evolved solution corresponds to setting $t = \infty$ in (4.1). Thus the fully evolved deflection under the centre of the circular load is

$$\begin{aligned}\eta_\infty &= -\frac{P_0}{\pi\rho R}\int_0^\infty \frac{\tanh(kH)J_1(kR)}{c(k)} \int_0^\infty J_0(kVs)\sin(kcs)\mathrm{d}s\mathrm{d}k \\ &= -\frac{P_0}{\pi\rho R}\int_0^\infty \frac{\tanh(kH)J_1(kR)}{c(k)} \frac{U(c-V)}{[k^2(c^2-V^2)]^{1/2}}\mathrm{d}k;\end{aligned}$$

and on introducing $a = kc^2/[g\tanh(kH)]$ and $b = kV^2/[g\tanh(kH)]$ we obtain the integral form derived by Nevel (1970):

$$\eta_\infty = -\frac{P_0}{\pi\rho R}\int_0^\infty \frac{J_1(kR)}{R}\,\frac{U(a-b)}{[a(a-b)]^{1/2}}\mathrm{d}k$$
$$= -\frac{P_0}{\pi\rho g\ell^2}\int_0^\infty \frac{J_1(\xi R/\ell)}{R/\ell}\,\frac{U(a-b)}{[a(a-b)]^{1/2}}\mathrm{d}\xi, \tag{4.2}$$

where $U$ is the Heaviside unit step function introduced earlier and $\xi = kl$ is the non-dimensional wavenumber (where $l = (D/\rho g)^{1/4}$ is a characteristic length). Non-zero contributions to the integral correspond to wavenumbers $k$ where $V < c(k)$, or equivalently $a > b$. Nevel noted that his integral in (4.2) is unbounded when $a = b$ and $\mathrm{d}a/\mathrm{d}\xi = \mathrm{d}b/\mathrm{d}\xi$, which in the deep water limit where $\tanh(kH) \approx 1$ corresponds to

$$V \approx 2\left(\frac{Dg^3}{27\rho}\right)^{1/2},$$

the critical speed originally identified by Kheysin (1967) that is equivalent to $c_{min}$.

We could numerically evaluate the deflection given by (4.1) at various times $t$, to examine the evolution of the response for any given load speed $V$, including this critical speed. However, we are content to evaluate the integral in (4.1) asymptotically (as $t \to \infty$) in § 6, since the results obtained generally confirm the asymptotic analysis for the concentrated point load given in the next section.

## 5. Large-time asymptotic analysis for the point load

Rather than asymptotically estimating the deflection $\eta$ for large $t$ directly, we can often simplify our calculation by first differentiating with respect to $t$ and estimating $\eta_t$ instead, since the integrand is a continuous function of $k$ and $\theta$. We use the method of stationary phase (Nayfeh 1981; Lighthill 1978). Thus taking the derivative of (3.7) with respect to $t$, and retaining only the component which can contribute asymptotically owing to points of stationary phase, we consider

$$\eta_t(r,\xi,t) = \frac{P_0}{4\pi^2\rho}\mathrm{Im}\left(\int_0^\infty\int_{-\pi/2}^{\pi/2}\frac{\mathscr{C}(r,\theta-\xi;k)}{c(k)}\mathrm{e}^{-\mathrm{i}\Psi t}\mathrm{d}\theta\mathrm{d}k\right) \tag{5.1}$$

where $\Psi = k(c - V\cos\theta)$ and $\mathrm{Im}(z)$ denotes the imaginary part of $z$ and

$$\mathscr{C}(r,\xi;k) = \frac{k\tanh(kH)\mathrm{e}^{\mathrm{i}kr\cos\xi}}{c(k)}.$$

We can then integrate our asymptotic results for $\eta_t$, to get asymptotic estimates (as $t \to \infty$) for the deflection $\eta$ (Olver 1974).

Major contributions to the integral come from the neighbourhood of points of stationary phase. The stationary points of the double integral correspond to the $(k,\theta)$ pairs satisfying the two equations $\Psi_k(k,\theta) = c_g - V\cos\theta = 0$ and $\Psi_\theta(k,\theta) = kV\sin\theta = 0$ (with subscripts $k$ and $\theta$ indicating partial differentiation), where $c_g = \mathrm{d}\omega/\mathrm{d}k$ denotes the group speed (Jones & Kline 1958; Cooke 1982).

Note that the stationary points depend crucially on the load speed $V$. Unless $k = 0$, they correspond to $\theta = 0$. In figure 1 for example, we depict the zeros of $\Psi_k$ (the two points of intersection $k_A$, $k_B$ of the ordinate representing load speed $V$ with the $c_g$-curve) when $c_{min} < V < (gH)^{1/2}$. For all load speeds $V < (gH)^{1/2}$, any stationary point is an interior point. For the load speed $V = (gH)^{1/2}$, one stationary point is on the boundary (at the origin) and the other is an interior point. Considering the integration to be over the semi-infinite strip in the Cartesian $(k,\theta)$-plane where $|\theta| \leqslant \frac{1}{2}\pi$ and $k \geqslant 0$, when $k = 0$ we have $c_g = (gH)^{1/2}$ so there are likewise stationary points on the boundary of this semi-infinite strip where $\theta = \pm\cos^{-1}[(gH)^{1/2}/V]$, for $V > (gH)^{1/2}$.

We now discuss the asymptotic contribution to the double integral in (5.1) from the neighbourhood of these stationary points for various load speeds $V$, in a fashion similar to the discussion given by Schulkes & Sneyd (1988) for the single integral in the case of a line load, cf. also Squire *et al.* (1996). Note that here it is the second- or some higher-order partial derivative in the phase function $\Psi = k(c - V\cos\theta)$ that is non-zero at the stationary point, in each case.

*Case 1: Subcritical load speeds $V \leqslant c_{gmin}$*

*Case 1a*: $V < c_{gmin}$

If the load speed $V$ is less than $c_{gmin}$, the minimum value of the group speed, then $\Psi$ is a monotonically increasing function of $k$ for any $\theta$ so there are no real stationary points. It follows that the transient deflection decays exponentially with time, hence the steady state is approached relatively rapidly.

*Case 1b*: $V = c_{gmin}$

When $V = c_{gmin}$, there is only a single stationary point, namely a point of inflexion relative to $k$, which we denote by $(k_s, 0)$. The Taylor expansion of the phase function $\Psi$ in (5.1) about this point is thus

$$\Psi(k,\theta) = \Psi(k_s,0) + \frac{\Psi_{kkk}}{6}(k-k_s)^3 + \frac{\Psi_{\theta\theta k}}{2}\theta^2(k-k_s) + \frac{\Psi_{\theta\theta}}{2}\theta^2 + \cdots \tag{5.2}$$

where the partial derivatives are evaluated at the stationary point. We have that $\Psi_{kkk}(k_s,0) = (c_g)_{kk}(k_s,0) > 0$, $\Psi_{\theta\theta}(k_s,0) = k_s V > 0$, and $\Psi_{\theta\theta k}(k_s,0) = V > 0$, so introducing (5.2) into (5.1) gives

$$\begin{aligned}
&\eta_t(r,\xi,t) \\
&\approx \frac{P_0}{4\pi^2\rho}\mathrm{Im}\left(\int_{k_s-\epsilon}^{k_s+\epsilon}\int_{-\epsilon}^{\epsilon}\mathscr{C}(r,\xi;k_s)\exp\left[-\mathrm{i}t(\Psi(k_s,0) + \frac{(c_g)_{kk}}{6}(k-k_s)^3 + \frac{kV}{2}\theta^2)\right]\mathrm{d}\theta\mathrm{d}k\right) \\
&= \frac{P_0}{2\pi^2\rho}\mathrm{Im}\left(\mathscr{C}(r,\xi;k_s)\mathrm{e}^{-\mathrm{i}t\Psi(k_s,0)}\int_0^\infty\exp\left[-\mathrm{i}t\frac{(c_g)_{kk}}{6}(k-k_s)^3\right]\left(\int_0^\infty\exp\left[-\mathrm{i}t\frac{V}{2}k\theta^2\right]\mathrm{d}\theta\right)\mathrm{d}k\right),
\end{aligned} \tag{5.3}$$

since $\epsilon > 0$ can be replaced by $\infty$ in the limit $t \to \infty$ and the integrand is even in $\theta$. To the leading-order term, the integral can be evaluated to yield

$$\eta_t(r,\xi,t) \approx \frac{P_0\sqrt{3}}{6\rho(\pi^3 k_s)^{1/2}}\Gamma(\tfrac{1}{3})\mathrm{Im}\left(\frac{\mathscr{C}(r,\xi;k_s)\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2V((c_g)_{kk}/6)^{1/3}}}\frac{\mathrm{e}^{-\mathrm{i}t\Psi(k_s,0)}}{t^{5/6}}\right). \tag{5.4}$$

We now integrate (5.4) over $T < t < \infty$ where $T$ is a sufficiently large fixed time (Olver 1974). Then after integrating by parts twice, we obtain (as $t \to \infty$)

$$\eta(r,\xi,t) - \eta(r,\xi,\infty) \approx \frac{P_0\sqrt{3}}{6\rho(\pi^3 k_s)^{1/2}}\Gamma\left(\tfrac{1}{3}\right)\mathrm{Im}\left(\frac{\mathscr{C}(r,\xi;k_s)\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2V((c_g)_{kk}/6)^{1/3}}}\right.$$
$$\left.\times\left[\frac{\mathrm{e}^{-\mathrm{i}t\Psi(k_s,0)}}{\mathrm{i}\Psi(k_s,0)t^{5/6}}\left(1+\frac{5\mathrm{i}}{6\Psi(k_s,0)t}\right) - \frac{55}{36\Psi(k_s,0)^2}\int_t^\infty \frac{\mathrm{e}^{-\mathrm{i}\tau\Psi(k_s,0)}}{\tau^{17/6}}\mathrm{d}\tau\right]\right). \quad (5.5)$$

For a concentrated line load at this load speed ($V = c_{gmin}$), Schulkes & Sneyd (1988) found that the most persistent transient decay is $t^{-1/3}$, whereas from (5.5) we conclude that transients due to a point load decay more quickly (as $t^{-5/6}$). Thus the deflection produced by an impulsively started concentrated point load tends rather more rapidly to the eventual steady state, by an additional factor $t^{-1/2}$.

*Case 2: Subcritical load speeds $c_{gmin} < V < c_{min}$ and supercritical load speeds $c_{min} < V < (gH)^{1/2}$*

In these two load speed regimes, the two interior stationary points are denoted by $(k_A, 0)$ and $(k_B, 0)$, as illustrated in figure 1 for the supercritical regime $c_{min} < V < (gH)^{1/2}$. After expanding the integrand about the stationary points, the integral becomes

$$\eta_t(r,\xi,t) \approx \frac{P_0}{4\pi^2\rho}\mathrm{Im}\left(\int_{k_A-\epsilon}^{k_A+\epsilon}\int_{-\epsilon}^{\epsilon}\mathscr{C}(r,\xi;k_A)\exp\left[-\mathrm{i}t\left(\Psi_A+\frac{\Psi_{kk}}{2}(k-k_A)^2+\frac{\Psi_{\theta\theta}}{2}\theta^2\right)\right]\mathrm{d}\theta\mathrm{d}k\right.$$
$$\left.+\int_{k_B-\epsilon}^{k_B+\epsilon}\int_{-\epsilon}^{\epsilon}\mathscr{C}(r,\xi;k_B)\exp\left[-\mathrm{i}t\left(\Psi_B+\frac{\Psi_{kk}}{2}(k-k_B)^2+\frac{\Psi_{\theta\theta}}{2}\theta^2\right)\right]\mathrm{d}\theta\mathrm{d}k\right) \quad (5.6)$$

where the derivatives $\Psi_{kk}$ and $\Psi_{\theta\theta}$ are evaluated at the stationary points. We observe that $\Psi_{kk}(k_B,0) > 0$, $\Psi_{kk}(k_A,0) < 0$, $\Psi_{\theta\theta}(k_A,0) > 0$, $\Psi_{\theta\theta}(k_B,0) > 0$ (and $\Psi_{k\theta} = \Psi_{\theta k} = 0$ at both $k_A$ and $k_B$). We also write $\Psi_A$ to denote $\Psi(k_A,0)$ and $\Psi_B$ to denote $\Psi(k_B,0)$.

Further evaluation as before yields

$$\eta_t(r,\xi,t) \approx \frac{P_0}{4\pi^2\rho}\mathrm{Im}\left(\mathscr{C}(r,\xi;k_A)\mathrm{e}^{-\mathrm{i}t\Psi_A}\int_{-\infty}^{\infty}\mathrm{e}^{-S^2}\frac{\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}}{t^{1/2}[|\Psi_{kk}(k_A,0)|]^{1/2}}\mathrm{d}S\right.$$
$$\left.\times\int_{-\infty}^{\infty}\mathrm{e}^{-U^2}\frac{\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}}{t^{1/2}[\Psi_{\theta\theta}(k_A,0)]^{1/2}}\mathrm{d}U\right)$$
$$+\frac{P_0}{4\pi^2\rho}\mathrm{Im}\left(\mathscr{C}(r,\xi;k_B)\mathrm{e}^{-\mathrm{i}t\Psi_B}\int_{-\infty}^{\infty}\mathrm{e}^{-S^2}\frac{\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}}{t^{1/2}[\Psi_{kk}(k_B,0)]^{1/2}}\mathrm{d}S\right.$$
$$\left.\times\int_{-\infty}^{\infty}\mathrm{e}^{-U^2}\frac{\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}}{t^{1/2}[\Psi_{\theta\theta}(k_B,0)]^{1/2}}\mathrm{d}U\right)$$
$$=\frac{P_0}{2\pi\rho}\mathrm{Im}\left(\frac{\mathscr{C}(r,\xi;k_A)}{[|\Psi_{kk}(k_A,0)|]^{1/2}[\Psi_{\theta\theta}(k_A,0)]^{1/2}}\frac{\mathrm{e}^{-\mathrm{i}t\Psi_A}}{t}\right)$$
$$-\frac{P_0}{2\pi\rho}\mathrm{Re}\left(\frac{\mathscr{C}(r,\xi;k_B)}{[\Psi_{kk}(k_B,0)]^{1/2}[\Psi_{\theta\theta}(k_B,0)]^{1/2}}\frac{\mathrm{e}^{-\mathrm{i}t\Psi_B}}{t}\right). \quad (5.7)$$

We again integrate over $T < t < \infty$ where $T$ is a sufficiently large fixed time, to obtain (after integrating by parts twice)

$$\begin{aligned}\eta(r,\xi,t)-\eta(r,\xi,\infty) \approx \frac{P_0}{2\pi\rho}\mathrm{Im}\Bigg(&\frac{\mathscr{C}(r,\xi;k_A)}{[|\Psi_{kk}(k_A,0)|]^{1/2}[\Psi_{\theta\theta}(k_A,0)]^{1/2}}\\ &\times\left[\frac{\mathrm{e}^{-\mathrm{i}t\Psi_A}}{\mathrm{i}\Psi_A t}\left(1+\frac{\mathrm{i}}{\Psi_A t}\right)-\frac{2}{(\Psi_A)^2}\int_t^\infty \frac{\mathrm{e}^{-\mathrm{i}\tau\Psi_A}}{\tau^3}\mathrm{d}\tau\right]\Bigg)\\ &-\frac{P_0}{2\pi\rho}\mathrm{Re}\Bigg(\frac{\mathscr{C}(r,\xi;k_B)}{[\Psi_{kk}(k_B,0)]^{1/2}[\Psi_{\theta\theta}(k_B,0)]^{1/2}}\\ &\times\left[\frac{\mathrm{e}^{-\mathrm{i}t\Psi_B}}{\mathrm{i}\Psi_B t}\left(1+\frac{\mathrm{i}}{\Psi_B t}\right)-\frac{2}{(\Psi_B)^2}\int_t^\infty \frac{\mathrm{e}^{-\mathrm{i}\tau\Psi_B}}{\tau^3}\mathrm{d}\tau\right]\Bigg). \quad (5.8)\end{aligned}$$

Equation (5.8) shows that, in the two load speed regimes discussed in this section, the transients due to a concentrated point load decay as $t^{-1}$. This decay rate is again faster by a factor $t^{-1/2}$, compared with the case of a concentrated line load discussed by Schulkes & Sneyd (1988).

*Case 3: Supercritical load speeds $V > (gH)^{1/2}$*

When the load speed $V$ is greater than $(gH)^{1/2}$, the stationary points near the origin are given by $(0, \pm\cos^{-1}[(gH)^{1/2}/V])$. The third stationary point away from the origin is given by $(k_B, 0)$. At the stationary point $(0,\theta_s)$ where $\theta_s = \cos^{-1}[(gH)^{1/2}/V]$, the non-zero derivatives up to the third order are $\Psi_{k\theta} = \Psi_{\theta k} = V\sin\theta_s$, $\Psi_{kkk} = (c_g)_{kk} < 0$, $\Psi_{\theta\theta k} = V\cos\theta_s$, so we get

$$\begin{aligned}&\eta_t(r,\xi,t)\\ &\approx \frac{P_0}{4\pi^2\rho}\mathrm{Im}\Bigg(\int_0^\epsilon\int_{\theta_s-\epsilon}^{\theta_s+\epsilon}\frac{H}{c(0)}k^2\exp\left[-\mathrm{i}t\left(\Psi_{k\theta}k(\theta-\theta_s)+\frac{\Psi_{\theta\theta k}}{2}k(\theta-\theta_s)^2+\frac{\Psi_{kkk}}{6}k^3\right)\right]\mathrm{d}\theta\mathrm{d}k\Bigg)\\ &\quad+\frac{P_0}{4\pi^2\rho}\mathrm{Im}\Bigg(\int_0^\epsilon\int_{-\theta_s-\epsilon}^{-\theta_s+\epsilon}\frac{H}{c(0)}k^2\\ &\quad\times\exp\left[-\mathrm{i}t\left(-\Psi_{k\theta}k(\theta+\theta_s)+\frac{\Psi_{\theta\theta k}}{2}k(\theta+\theta_s)^2+\frac{\Psi_{kkk}}{6}k^3\right)\right]\mathrm{d}\theta\mathrm{d}k\Bigg)\\ &\quad+\frac{P_0}{4\pi^2\rho}\mathrm{Im}\Bigg(\int_{k_B-\epsilon}^{k_B+\epsilon}\int_{-\epsilon}^{\epsilon}\mathscr{C}(r,\xi;k_B)\exp\left[-\mathrm{i}t\left(\Psi_B+\frac{\Psi_{kk}}{2}(k-k_B)^2+\frac{\Psi_{\theta\theta}}{2}\theta^2\right)\right]\mathrm{d}\theta\mathrm{d}k\Bigg).\end{aligned} \quad (5.9)$$

In the limit $t \to \infty$, we again replace $\epsilon > 0$ by $\infty$, and evaluate the last integral in (5.9) as in the previous cases. The first two integrals in (5.9) can be combined and the inner $\theta$ integral then evaluated, so we obtain

$$\begin{aligned}\eta_t(r,\xi,t) \approx \frac{P_0}{\pi^{3/2}\rho}\mathrm{Im}\Bigg(&\frac{H}{c(0)}\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{[2V\cos\theta_s]^{1/2}t^{1/2}}\int_0^\infty k^{3/2}\\ &\times\exp\left[\mathrm{i}t\left(\frac{V}{2}\cos\theta_s\tan^2(\theta_s)k-\frac{(c_g)_{kk}}{6}k^3\right)\right]\mathrm{d}k\Bigg)\\ &-\frac{P_0}{2\pi\rho}\mathrm{Re}\left(\frac{\mathscr{C}(r,\xi;k_B)}{[\Psi_{kk}(k_B,0)]^{1/2}[\Psi_{\theta\theta}(k_B,0)]^{1/2}}\frac{\mathrm{e}^{-\mathrm{i}t\Psi_B}}{t}\right). \quad (5.10)\end{aligned}$$

The integral in (5.10) can be estimated using the method of steepest descent (see Nugroho 1997 for details). Thus we obtain

$$\eta_t(r,\xi,t) \approx \frac{P_0}{2\pi\rho}\left(\frac{H\tan\theta_s}{c(0)|(c_g)_{kk}|}\frac{\exp\left[\dfrac{-\tan^3(\theta_s)}{3}\left(\dfrac{(V\cos\theta_s)^3}{|(c_g)_{kk}|}t\right)\right]^{1/2}}{t}\right)$$

$$-\frac{P_0}{2\pi\rho}\mathrm{Re}\left(\frac{\mathscr{C}(r,\xi;k_B)}{[\Psi_{kk}(k_B,0)]^{1/2}[\Psi_{\theta\theta}(k_B,0)]^{1/2}}\frac{\mathrm{e}^{-\mathrm{i}t\Psi_B}}{t}\right). \tag{5.11}$$

We again integrate over $T < t < \infty$ where $T$ is a sufficiently large fixed time, to obtain

$$\eta(r,\xi,t)-\eta(r,\xi,\infty)\approx\frac{P_0}{2\pi\rho}\left(\frac{H\tan\theta_s}{c(0)[|(c_g)_{kk}|]^{1/2}}\left[\frac{3\exp\left[-\dfrac{(\tan\theta_s)^3}{3}\left[\dfrac{(V\cos\theta_s)^3}{|(c_g)_{kk}|}\right]^{1/2}\right]}{[(V\cos\theta_s)^3]^{1/2}(\tan\theta_s)^3 t}\right.\right.$$

$$\times\left(1-\frac{3[|(c_g)_{kk}|]^{1/2}}{[(V\cos\theta_s)^3]^{1/2}(\tan\theta_s)^3 t}\right)$$

$$\left.\left.+\frac{18|(c_g)_{kk}|}{(V\cos\theta_s)^3(\tan\theta_s)^6}\int_t^\infty\frac{\exp\left[-\dfrac{2}{3}\left[\dfrac{(V\cos\theta_s)^3}{4|(c_g)_{kk}|}\right]^{1/2}(\tan\theta_s)^3\tau\right]}{\tau^3}\mathrm{d}\tau\right]\right)$$

$$-\frac{P_0}{2\pi\rho}\mathrm{Re}\left(\frac{\mathscr{C}(r,\xi;k_B)}{[\Psi_{kk}(k_B,0)]^{1/2}[\Psi_{\theta\theta}(k_B,0)]^{1/2}}\left[\frac{\mathrm{e}^{-\mathrm{i}t\Psi_B}}{\mathrm{i}\Psi_B t}\left(1+\frac{\mathrm{i}}{\Psi_B t}\right)\right.\right.$$

$$\left.\left.-\frac{2}{(\Psi_B)^2}\int_t^\infty\frac{\mathrm{e}^{-\mathrm{i}\tau\Psi_B}}{\tau^3}\mathrm{d}\tau\right]\right). \tag{5.12}$$

Hence waves behind the load, which correspond to the contribution from the stationary points $(0, \pm\theta_s)$, disappear exponentially, i.e. much faster than the waves in front. The contribution from the stationary point $(k_B, 0)$ defines the slower decay (as $t^{-1}$) of the transients associated with the leading shorter flexural waves. Once again, this decay is faster by a factor $t^{-1/2}$ compared with the case of a concentrated line load discussed by Schulkes & Sneyd (1988).

*Case 4: Critical load speed $V = c_{min}$*

For the critical load speed $V = c_{min}$, Schulkes & Sneyd (1988) showed that the deflection grows in time as $t^{1/2}$, consistent with Kheysin (1971). For our double integral, at this load speed there are again two interior stationary points which we denote by $(k_A, 0)$ and $(k_{min}, 0)$, where $k_{min}$ is the wavenumber at $c_{min}$. Since $\Psi(k_{min}, 0) = 0$, the Taylor expansion of the phase function $\Psi(k,\theta)$ about $k = k_{min}$ is

$$\Psi(k,\theta) = \frac{\Psi_{kk}}{2}(k-k_{min})^2 + \frac{\Psi_{\theta\theta}}{2}\theta^2 + \cdots,$$

and we note that $\Psi_{kk}(k_A, 0) < 0$, $\Psi_{kk}(k_{min}, 0) > 0$, $\Psi_{\theta\theta}(k_A, 0) > 0$, $\Psi_{\theta\theta}(k_{min}, 0) > 0$. Thus we obtain

$$\eta_t(r, \xi, t) \approx \frac{P_0}{2\pi\rho} \mathrm{Im} \left( \frac{\mathscr{C}(r, \xi\,; k_A)}{[|\Psi_{kk}(k_A, 0)|]^{1/2} [\Psi_{\theta\theta}(k_A, 0)]^{1/2}} \frac{\mathrm{e}^{-\mathrm{i}t\Psi_A}}{t} \right) - \frac{P_0}{2\pi\rho} \mathrm{Re} \left( \frac{\mathscr{C}(r, \xi\,; k_{min})}{[\Psi_{kk}(k_{min}, 0)]^{1/2} [\Psi_{\theta\theta}(k_{min}, 0)]^{1/2}} \frac{1}{t} \right). \tag{5.13}$$

Rather than integrating from $t$ to $\infty$, in this case we integrate from $T$ to $t$ where $T$ is a sufficiently large fixed time, to get

$$\eta(r, \xi, t) - \eta(r, \xi, T) \approx \frac{P_0}{2\pi\rho} \mathrm{Im} \left( \frac{\mathscr{C}(r, \xi\,; k_A)}{[|\Psi_{kk}(k_A, 0)|]^{1/2} [\Psi_{\theta\theta}(k_A, 0)]^{1/2}} \times \left[ \frac{\mathrm{e}^{-\mathrm{i}t\Psi_A}}{\mathrm{i}\Psi_A t} \left( 1 + \frac{\mathrm{i}}{\Psi_A t} \right) + \frac{\mathrm{e}^{-\mathrm{i}T\Psi_A}}{\mathrm{i}\Psi_A T} \left( 1 + \frac{\mathrm{i}}{\Psi_A T^2} \right) - \frac{2}{(\Psi_A)^2} \int_T^t \frac{\mathrm{e}^{-\mathrm{i}\tau\Psi_A}}{\tau^3} \mathrm{d}\tau \right] \right) - \frac{P_0}{2\pi\rho} \mathrm{Re} \times \left( \frac{\mathscr{C}(r, \xi\,; k_{min})}{[\Psi_{kk}(k_{min}, 0)]^{1/2} [\Psi_{\theta\theta}(k_{min}, 0)]^{1/2}} (\ln t - \ln T) \right). \tag{5.14}$$

Thus the deflection grows logarithmically in the limit $t \to \infty$. This is similar to the case of a concentrated line load, in that there is no steady solution at the critical load speed $c_{min}$ (Kheysin 1971; Schulkes & Sneyd 1988), but the growth rate predicted for the concentrated point load is $O(\ln t)$ rather than $O(t^{1/2})$.

*Case 5: Load speed* $V = (gH)^{1/2}$

As $V \to (gH)^{1/2}$ from below, the stationary point $(k_A, 0)$ nearer to the origin $(0, 0)$ approaches it along $\theta = 0$, so in the limit $(k_A, 0) = (0, 0)$ is a boundary stationary point. When $V \to (gH)^{1/2}$ from above, the two interior stationary points $[0, \pm\cos^{-1}(gH)^{1/2}/V]$ merge into the origin. In passing, we recall that Schulkes & Sneyd (1988) noted that their relevant phase function, corresponding to setting $\theta = 0$ in our $\Psi = k(c - V\cos\theta)$, has a triple zero. At the origin, in our analysis we have that all partial derivatives of $\Psi$ up to second order vanish and $\Psi_{kkk} = (c_g)_{kk} < 0$, $\Psi_{\theta\theta\theta} = -kV\sin\theta = 0$, $\Psi_{\theta\theta k} = V$, and $\Psi_{\theta kk} = 0$. Once again, introducing Taylor expansions into (5.1) we obtain

$$\eta_t(r, \xi, t) \approx \frac{P_0}{2\pi^2\rho} \mathrm{Im} \left( \int_0^\epsilon \int_0^\epsilon \frac{H}{c(0)} k^2 \exp\left[ -\mathrm{i}t \left( \frac{(c_g)_{kk}}{6} k^3 + \frac{V}{2} k\theta^2 \right) \right] \mathrm{d}\theta \mathrm{d}k \right) + \frac{P_0}{4\pi^2\rho} \mathrm{Im} \left( \int_{k_B-\epsilon}^{k_B+\epsilon} \int_{-\epsilon}^{\epsilon} \mathscr{C}(r, \xi\,; k_B) \exp\left[ -\mathrm{i}t \left( \Psi_B + \frac{\Psi_{kk}}{2}(k - k_B)^2 + \frac{\Psi_{\theta\theta}}{2}\theta^2 \right) \right] \mathrm{d}\theta \mathrm{d}k \right), \tag{5.15}$$

where each of the derivatives is evaluated at the relevant stationary point. Further evaluation yields

$$\eta_t(r,\xi,t) \approx -\frac{P_0}{12(\pi^3)^{1/2}\rho}\frac{H}{c(0)}\left(\frac{6}{|(c_g)_{kk}|}\right)^{5/6}\frac{\Gamma(\frac{5}{6})}{(2V)^{1/2}}\frac{1}{t^{4/3}} - \frac{P_0}{2\pi\rho}\mathrm{Re}\left(\frac{\mathscr{C}(r,\xi;k_B)}{[\Psi_{kk}(k_B,0)]^{1/2}[\Psi_{\theta\theta}(k_B,0)]^{1/2}}\frac{\mathrm{e}^{-\mathrm{i}t\Psi_B}}{t}\right).$$

We again integrate over $T < t < \infty$ for sufficiently large fixed $T$, to obtain (as $t \to \infty$)

$$\eta(r,\xi,t) - \eta(r,\xi,\infty) \approx \frac{P_0}{4(\pi^3)^{1/2}\rho}\frac{H}{c(0)}\left(\frac{6}{|(c_g)_{kk}|}\right)^{5/6}\frac{\Gamma(\frac{5}{6})}{(2V)^{1/2}}\frac{1}{t^{1/3}} - \frac{P_0}{2\pi\rho}\mathrm{Re}\left(\frac{\mathscr{C}(r,\xi;k_B)}{[\Psi_{kk}(k_B,0)]^{1/2}[\Psi_{\theta\theta}(k_B,0)]^{1/2}} \times\left[\frac{\mathrm{e}^{-\mathrm{i}t\Psi_B}}{\mathrm{i}\Psi_B t}\left(1+\frac{\mathrm{i}}{\Psi_B t}\right) - \frac{2}{(\Psi_B)^2}\int_t^\infty \frac{\mathrm{e}^{-\mathrm{i}\tau\Psi_B}}{\tau^3}\mathrm{d}\tau\right]\right). \quad (5.16)$$

The transient due to the stationary point $(0,0)$ decays as $t^{-1/3}$, whereas that due to the stationary point $(k_B,0)$ decays more quickly (as $t^{-1}$). This behaviour is in contrast to that for load speed $V = (gH)^{1/2}$ in the case of a concentrated line load, where Schulkes & Sneyd (1988) found that the deflection grows as $t^{1/3}$. A growing deflection has not been observed when $V = (gH)^{1/2}$ (see Takizawa 1985; Squire *et al.* 1985); and Schulkes & Sneyd (1988) noted 'It is possible that $V = (gH)^{1/2}$ does not represent a critical speed for *two-dimensional sources*, because wave energy could radiate in all directions – not just along the line of motion'. Our result shows that the time-dependent response is transient for a concentrated point load, so that the eventual deflection at $V = (gH)^{1/2}$ is steady state. This is consistent with Milinazzo *et al.* (1995), who found that the deflection due to a uniformly moving distributed rectangular load is finite at the load speed $(gH)^{1/2}$.

## 6. Large-time asymptotic analysis for the distributed circular load

The time-dependent response under the centre of a uniformly distributed circular load is clarified by introducing the dimensionless time $t^* = kVt$, so that for the inner integral of (4.1) we have

$$\int_0^{t^*} J_0(s)\sin(\lambda s)\mathrm{d}s = \frac{1}{\pi}\int_0^{t^*}\int_0^{\pi}\cos(s\sin\theta)\mathrm{d}\theta\sin(\lambda s)\mathrm{d}s = \mathrm{Re}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-\mathrm{e}^{\mathrm{i}t^*(\lambda-\sin\theta)}}{\lambda-\sin\theta}\mathrm{d}\theta\right) \quad (6.1)$$

where $\lambda = c(k)/V$. Noting that

$$\int_{-\pi}^{\pi}\frac{1}{\lambda-\sin\theta}\mathrm{d}\theta = \begin{cases} 2\pi/(\lambda^2-1)^{1/2}, & \lambda^2 > 1 \\ 0, & \lambda^2 \leqslant 1 \end{cases}$$

and returning to the original notation of (4.1), as $t \to \infty$ we therefore have

$$\int_0^t J_0(kVs)\sin(kcs)\mathrm{d}s \approx \frac{U(c-V)}{[k^2(c^2-V^2)]^{1/2}} - \mathrm{Re}\left\{\frac{1}{[2\pi kVt]^{1/2}}\frac{\mathrm{e}^{-\mathrm{i}[k(c-V)t+\pi/4]}}{k(c-V)}\right\} \quad (6.2)$$

for all wavenumbers $k$ except the one or two values where $c(k) = V$ (see figure 1). Substituting into (4.1), the time-independent component on the right-hand side of (6.2) produces the result (4.2) due to Nevel (1970) outlined in § 4. The time-dependent component on the right-hand side of (6.2) corresponds to the dominant time-dependent contribution (as $t \to \infty$) to the integral on the right-hand side of (6.1), arising from the neighbourhood of the point of stationary phase where $\cos\theta = 0$ (i.e. $\theta = \pi/2$). Thus from (4.1) we have the dominant time-dependent contribution to the deflection (as $t \to \infty$), supplementing the evolved ($t = \infty$) solution (4.2) due to Nevel (1970), in the form

$$\tilde{\eta}(t) = \frac{P_0}{\rho R}\frac{1}{[2\pi^3 V t]^{1/2}}\mathrm{Re}\int_0^\infty \frac{\tanh(kH)J_1(kR)}{k^{1/2}c(k)}\frac{\mathrm{e}^{-\mathrm{i}[k(c-V)t+\pi/4]}}{k(c-V)}\mathrm{d}k. \tag{6.3}$$

The integral in (6.3) resembles the integral $I_1$ analysed by Schulkes & Sneyd (1988) for a line load, except that the range of integration is $(0,\infty)$ and the integrand has additional entries that are unimportant asymptotically, namely $k^{-1/2}$ and the Bessel function $J_1(kR)$ corresponding to the uniformly distributed circular load of radius $R$. Accordingly, the transients they found in the vicinity of a one-dimensional line load are moderated by the factor $t^{-1/2}$ outside the integral in (6.3). Thus we find the transient decay rates produced by the distributed load precisely the same as for the concentrated point load given in the previous section, for the respective subcritical and supercritical load speed regimes, namely enhanced exponential decay if $V < c_{gmin}$, but $O(t^{-5/6})$ decay if $V = c_{gmin}$; and $O(t^{-1})$ decay if $c_{gmin} < V < c_{min}$ or $c_{min} < V < (gH)^{1/2}$ or $V > (gH)^{1/2}$. Consequently, the deflection produced by an impulsively started uniformly distributed circular load likewise tends rather more rapidly to the evolved steady state (in this case Nevel's solution) due to the factor $t^{-1/2}$, in comparison with the transients in these regimes for a concentrated line load discussed by Schulkes & Sneyd (1988).

Recall that for $c_{min} < V < (gH)^{1/2}$ there are two points of stationary phase $k = k_A, k_B\,(0 < k_A < k_B)$ for the integral in (6.3), as illustrated in figure 1, where the wave group speed coincides with the load speed ($c_g = V$). The dominant time-dependent contribution is produced in the neighbourhood of $k_A$ for the upper limiting speed $(gH)^{1/2}$, and in the neighbourhood of $k_B$ for the lower limiting speed $c_{min}$, the critical minimum phase speed of the hybrid waves. At these two load speeds, $(gH)^{1/2}$ and $c_{min}$, it is notable that the phase speed coincides with the group speed ($c = c_g$), so the points of stationary phase (at $k = 0$ and $k = k_{min}$ respectively) occur precisely where (6.2) and (6.3) are not valid.

Our previous analytical approach was to first investigate the asymptotic behaviour (as $t\to\infty$) of the time-derivative of the deflection and then deduce the corresponding asymptotic behaviour of the time-dependent component of the deflection. Thus from (4.1) we have the exact time-derivative

$$\eta'(t) = -\frac{P_0}{\pi\rho R}\int_0^\infty \frac{\tanh(kH)J_1(kR)}{c(k)}J_0(kVt)\sin(kct)\mathrm{d}k; \tag{6.4}$$

so invoking the well-known asymptotic form

$$J_0(x) \approx \left(\frac{2}{\pi x}\right)^{1/2}\cos(x-\pi/4), \quad x\to\infty, \tag{6.5}$$

for the Bessel function, we may first consider the behaviour of the asymptotic form for the time-derivative (as $t \to \infty$)

$$\eta'(t) = -\frac{P_0}{\rho R}\frac{1}{(2\pi^3 Vt)^{1/2}}\int_0^\infty \frac{\tanh(kH)J_1(kR)}{k^{1/2}c(k)}\sin[k(c-V)t+\pi/4]\mathrm{d}k, \tag{6.6}$$

rather than the asymptotic form for the deflection $\hat{\eta}(t)$ given by (6.3). In passing, we note that the asymptotic form in (6.6) also follows by differentiating the asymptotic form in (6.3), and it is an analytical advantage that the integral in (6.6) is regular at both critical load speeds [$c_{min}$ and $(gH)^{1/2}$].

When the load speed $V$ approaches the water wave speed $(gH)^{1/2}$ from below ($V \to (gH)^{1/2}-$), the dominant contribution to the integral in (6.3) arises in the decreasing small but finite interval $(0, k_A)$, since the origin is then the limiting value of the first point of stationary phase $k_A$. In particular, on neglecting the plate acceleration term the dispersion equation for propagated waves is (cf. Schulkes & Sneyd 1988; Squire *et al.* 1996)

$$c(k) = \left[\left(\frac{Dk^2}{\rho}+g\right)\frac{1}{k}\tanh(kH)\right]^{1/2} \approx (gH)^{1/2}\left(1-\frac{k^2H^2}{6}+\cdots\right), \tag{6.7}$$

so we have

$$c(k) - V \approx -\frac{k^2H^2}{6}V.$$

Hence when the load speed approaches the water wave speed from below [$V \to (gH)^{1/2}-$], from (6.6) we find $\eta'(t) = O(t^{-4/3})$ as $t\to\infty$, so that $\tilde{\eta}(t) = O(t^{-1/3})$ as $t\to\infty$. Thus we find transients that eventually die away relatively slowly at the centre of a uniformly distributed circular load, in contrast to the $O(t^{1/3})$ growth in the deflection found by Schulkes & Sneyd (1988) for an impulsively started line load. Of greater interest is the case of time-dependence at the critical load speed $V = c_{min}$, the minimum phase speed of propagated waves. In this case we write

$$c(k) = c_{min} + \tfrac{1}{2}c''(k_{min})(k-k_{min})^2 + \cdots,$$

where $k_{min}$ denotes the wave number corresponding to the minimum phase speed $c_{min}$ of generated waves. Thus as the load speed approaches this minimum phase speed from above ($V \to c_{min}+$), the dominant contribution to the integral in (6.6) arises in a decreasing but significant small interval $(k_{min}, k_B)$, where $k_B$ is the second point of stationary phase. We therefore find $\eta'(t) = O(t^{-1})$, so that $\eta(t) = O(\ln t)$ as $t\to\infty$. Thus the deflection due to an impulsively started uniform circular load eventually grows logarithmically with time, as we found for the impulsively started concentrated point load, in contrast with the $O(t^{1/2})$ growth as $t\to\infty$ found by Schulkes & Sneyd (1988) for an impulsively started line load.

We can also evaluate the asymptotic time-dependent response (as $t \to \infty$) directly from (4.1). Thus first exchanging the order of the integration and then introducing the asymptotic form (6.5) for the Bessel function, we get

$$\begin{aligned}\eta(t) &= -\frac{P_0}{\pi\rho R}\int_0^t\int_0^\infty \frac{\tanh(kH)J_1(kR)}{c(k)}J_0(kVs)\sin(kcs)\mathrm{d}k\mathrm{d}s \\ &\approx \eta(T) - \frac{P_0}{(2\pi^3VR)^{1/2}}\int_T^t \frac{1}{s^{1/2}}\int_0^\infty \frac{\tanh(kH)J_1(kR)}{k^{1/2}\,c(k)} \\ &\quad\times\big(\sin[k(c-V)s+\pi/4]+\sin[k(c+V)s-\pi/4]\big)\,\mathrm{d}k\mathrm{d}s\end{aligned} \tag{6.8}$$

for sufficiently large fixed $T > 0$. Consequently, we consider the stationary points for the integral

$$I(s) = \mathrm{Im}\left(\int_0^\infty \frac{\tanh(kH)J_1(kR)}{k^{1/2}c(k)} \mathrm{e}^{-\mathrm{i}k(c-V)s}\mathrm{d}k\right). \tag{6.9}$$

Thus, when $V \to (gH)^{1/2}$ from below and recalling (6.7), we have integral (6.9) represented asymptotically by

$$\int_0^\epsilon \frac{HR}{2V} k^{3/2} \mathrm{e}^{\mathrm{i}H^2Vk^3s/6}\mathrm{d}k = O(s^{-5/6}), \tag{6.10}$$

and hence the transient component of the deflection in (6.8) is

$$O\left(\int_T^t \frac{1}{s^{1/2}} s^{-5/6}\mathrm{d}s\right) = O(t^{-1/3}) \tag{6.11}$$

as before. In the neighbourhood of $k = k_{min}$, a similar asymptotic evaluation of integral (6.9) produces a result of $O(s^{-1/2})$, so that the growth rate as $V \to c_{min}$ from above is confirmed as

$$O\left(\int_T^t \frac{1}{s^{1/2}} s^{-1/2}\mathrm{d}s\right) = O(\ln t). \tag{6.12}$$

A similar analysis based upon (3.8) can be made for the concentrated point load, serving to emphasize that load concentration does not affect the time-dependence of the deflection underneath the centre of the load, and that this time-dependence is to be seen at any field position where the disturbance is measurable.

## 7. Numerical computation

To support the asymptotic results obtained in the previous sections, the double integral for the ice deflection $\eta$ in (3.6) was evaluated numerically using the fast Fourier transform (FFT), with from 512 to 4096 points in each direction. At the largest values of $t$, in some cases it was necessary to use the greatest resolution in order to obtain acceptable results. In figure 2($a$) the effect on the numerical results of increasing the resolution from 512 × 512 to 2048 × 2048 can be seen. The curves $\eta(18, 18, t)$ obtained using the FFT become smoother as the resolution increases. The curve corresponding to the 2048 × 2048 grid is smooth up to about $t = 170$.

Since the asymptotic estimates for $\eta$ are obtained by integrating $\eta_t$ with respect to $t$, it is necessary to account for the constant that comes from the integration before a comparison with the numerical results can be made. This is achieved by adding a vertical offset to the asymptotic estimate so that the asymptotic and numerical estimates agree at a fixed point in time. The offset thus corresponds to the prior or steady-state solution and as such depends on $(X, y)$.

Figure 2($a$–$e$) compares the numerical and asymptotic estimates of the values of $\eta(X, y, t)$ for various values of $(X, y)$. The parameter values are taken from Takizawa (1985). The particular points $(X, y)$ chosen have no special significance – the behaviour of the ice deflection at these points is representative (either ahead of or behind the load). In all cases, the agreement is very good.

Figure 3($a$–$c$) shows the evolution of the centreline profile of the ice deflection (i.e. along the line of motion of the load), for $V = 5.5, 7.0$, and $9.0\,\mathrm{m\,s^{-1}}$. These figures are oriented as in figure 3 of Takizawa (1985) – i.e. negative distance to the left corresponds to ahead of the load, and positive distance to the right corresponds to behind the load. We note that for $V = 5.5$ and $7.0\,\mathrm{m\,s^{-1}}$ the transients have largely

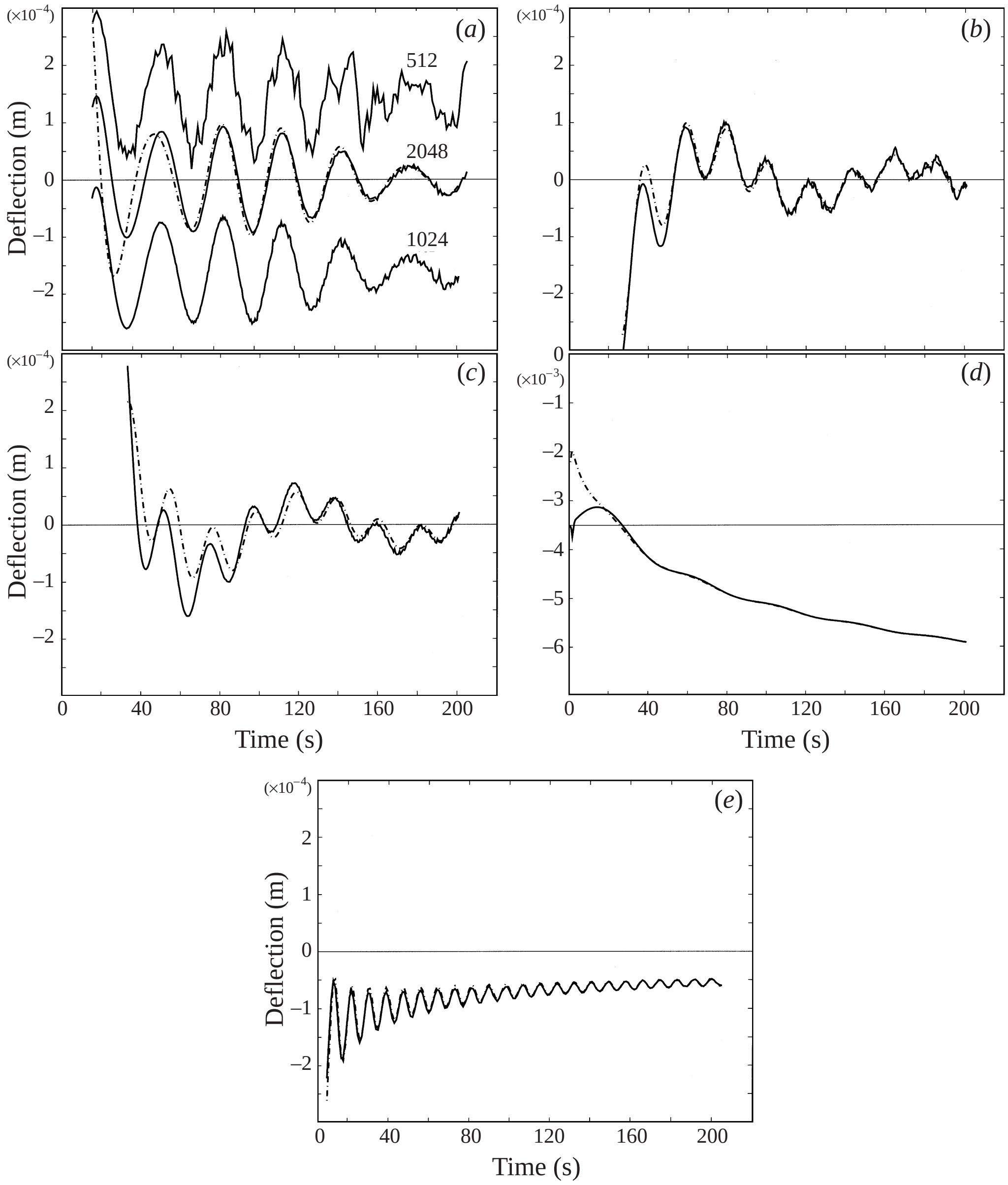


FIGURE 2. Deflection at various field points and load speeds: (*a*) ahead of the load at $X = 18.0$ m and $y = 18.0$ m and subcritical load speed $V = 5.5$ m s$^{-1}$; (*b*) behind the load at $X = -18.0$ m and $y = 18.0$ m and supercritical speed $V = 7.0$ m s$^{-1}$; (*c*) directly behind the load at $X = -50.0$ m and $y = 0.0$ m and the supercritical load speed $V = 7.0$ m s$^{-1}$; (*d*) directly behind the load at $X = -18.0$ m and $y = 0.0$ m and at the critical load speed $V = 6.2$ m s$^{-1}$ ($c_{min}$); (*e*) $X = 18.0$ m and $y = 0.0$ m at the load speed $(gH)^{1/2}$. The results obtained using the fast Fourier transform are shown as the solid curves ($512 \times 512$ grid to $2048 \times 2048$ grid in *a*, $1024 \times 1024$ in *b*, $2048 \times 2048$ in *c*,*d* and $4096 \times 4096$ in *e*), whereas the result obtained using the asymptotic approximation is shown as the dashed curve (equation (5.8) in *a*–*c*, (5.14) in *d* and (5.16) in *e*). The other parameters are $D = 2.5 \times 10^5$, $\nu = \frac{1}{3}$, $h = 0.175$ m, $H = 6.8$ m, $g = 9.8$ m s$^{-2}$.

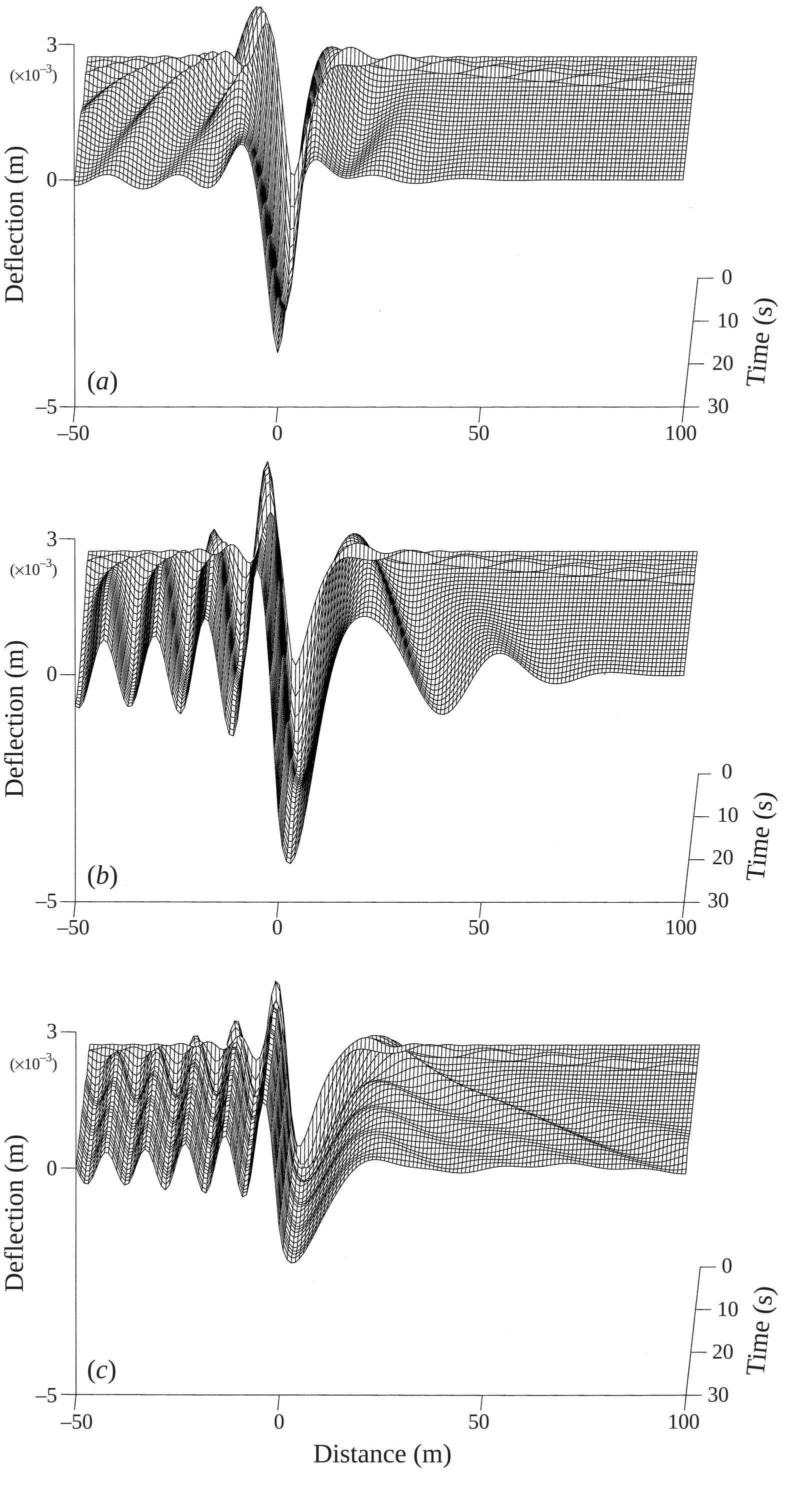


FIGURE 3. An illustration of the evolution of the centreline profile of the ice deflection for $0 \leqslant t \leqslant 30$ s and (*a*) $V = 5.5$ m s$^{-1}$, (*b*) $V = 7.0$ m s$^{-1}$, (*c*) $V = 9.0$ m s$^{-1}$. The results were obtained using the fast Fourier transform ($2048 \times 2048$ grid). The other parameters are $D = 2.5 \times 10^5$, $\nu = \frac{1}{3}$, $h = 0.175$ m, $H = 6.8$ m, $g = 9.8$ m s$^{-2}$.

disappeared by about $t = 20$, and for $V = 9.0\ \mathrm{m\,s^{-1}}$ the transients decay much more slowly.

## 8. Comparison with experiment

Squire *et al.* (1985) and Takizawa (1985) give experimental results that can be compared to the predictions made by the theory presented in this article. However, since the analytic results given in § 5 are for large time, it is necessary to make the comparison with results obtained by integrating (3.6) numerically.

In the experiments of Takizawa (1985), the amplification factor, defined to be the ratio of the maximum deflection at $V = c_{min}$ to that at $V = 0$, is found to be approximately 3. In those experiments, the vehicle travelled about 100 m at the speed $V = c_{min}$ before reaching the observation point. Assuming an instantaneous acceleration to the final speed, this corresponds to a travel time of 16.6 s. At $t = 16.6$ s, we compute an amplification factor of about 2.7. At $t = 30$ s, we compute an amplification factor of 3. It should be noted that for a point load moving steadily at speeds near $c_{min}$, Milinazzo *et al.* (1985) found an amplification value of approximately 3. For an impulsively started line load, Schulkes & Sneyd (1988) obtain a value of 6.6.

In addition, Takizawa (1985) observed that the position of the maximum depression lagged the position of the source. At $t = 16.6$ s we find that the lag is about 0.8 m for $V = c_{min}$ and 4 m for $V = 8$. In addition, we see that the lag increases rapidly with source speed for $V > c_{min}$. The corresponding observed experimental values are 3 m and 5 m respectively. For an impulsively started line load, Schulkes & Sneyd (1988) obtain values of one eighth of a wavelength for $V = c_{min}$ and one quarter of a wavelength for $V > c_{min}$. This translates to lags of between 2 and 3.5 m for the conditions that correspond to the experiments of Takizawa (1985).

Squire *et al.* (1985) measure an amplification of the ice stress, defined to be the ratio of the maximum stress at $V = c_{min}$ to that at $V = 0$, to be between 2.24 and 2.29 for sea ice and about 1.5 for lake ice. In that experiment the travel time was 32 s. Since the strain due to a point load is singular at the origin, for the purpose of comparison with experiment we replaced the point load with a load distributed on a rectangle of half-length $a$ and half-width $b$. The values of $a$ and $b$ were varied from 1.5 m to 3 m and 1.25 m to 2.5 m respectively. The computed amplifications are approximately 1 for lake ice and between 1.6 and 2 for sea ice. For an impulsively started line load, Schulkes & Sneyd (1988) give a value of about 5.

## 9. Conclusions

The steady-state deflection, caused by either a concentrated point load or a uniformly distributed circular load moving over a flexible floating plate, is generally reached more quickly in all load speed regimes than the time-dependent theory for a concentrated line load suggested. In the subcritical load speed regime $c_{gmin} \leqslant V < c_{min}$, and in the supercritical load speed regimes $c_{min} < V < (gH)^{1/2}$ and $V > (gH)^{1/2}$, the transient responses decay faster by a factor $t^{-1/2}$. Furthermore, a concentrated point load or a uniformly distributed circular load travelling at the critical speed $c_{min}$ produces a deflection that grows logarithmically with time (namely $O(\ln t)$ rather than $O(t^{1/2})$ as $t \to \infty$); but when travelling at the shallow water wave speed $(gH)^{1/2}$, there is a transient response that decays relatively slowly [namely $O(t^{-1/3})$]. All of these conclusions are independent of direction or distance from the load.

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## REFERENCES

Cooke, J. C. 1982 Stationary phase in two dimensions. *J. Inst. Maths Applics.* **29**, 29–37.

Davys, J. W., Hosking, R. J. & Sneyd, A. D. 1985 Waves due to a steadily moving source on a floating ice plate. *J. Fluid Mech.* **158**, 269–287.

Jones, D. S. & Kline, M. 1958 Asymptotic expansion of multiple integrals and the method of stationary phase. *J. Math. Phys.* **37**, 1–28.

Kerr, A. D. 1983 The critical velocities of a load moving on a floating ice plate that is subjected to in plane forces. *Cold Regions Sci. Technol.* **6**, 267–274.

Kerr, A. D. & Palmer, W. T. 1972 The deformation and stresses of floating ice plates. *Acta Mechanica* **15**, 57–72.

Kheysin, Ye 1967 *Dynamics of the Ice Cover*. Gidrometeorologicheskoe Izdat-el'stvo, Leningrad (technical transl. FSTC-HT-23-485-69, US Army Foreign Science and Technology Center).

Kheysin, Ye 1971 Some unsteady-state problems in ice-cover dynamics. *Studies in Ice Physics and Ice Engineering* (ed. G. N. Yakovlev), pp. 81–91. Israel Program for Scientific Translations.

Lighthill, M. J. 1978 *Waves In Fluids*. Cambridge University Press.

Milinazzo, F., Shinbrot, M. & Evans, N. W. 1995 A mathematical analysis of the steady response of floating ice to the uniform motion of a rectangular load. *J. Fluid Mech.* **287**, 173–197.

Nayfeh, A. H. 1981 *Introduction to Perturbation Techniques*. John Wiley & Sons.

Nevel, D. E. 1970 Moving loads on a floating ice sheet, *Cold Regions Research and Engineering Laboratory Res. Rep.* 261. Hanover, New Hampshire, USA.

Nugroho, W. S. 1997 Waves generated by a load moving on an ice plate over water. PhD Dissertation, University of Victoria, Canada.

Olver, F. W. J. 1974 *Introduction to Asymptotics and Special Functions*. Academic.

Schulkes, R. M. S. M. & Sneyd, A. D. 1988 Time-dependent response of floating ice plate to a steadily moving load. *J. Fluid Mech.* **186**, 25–46.

Squire, V. A., Hosking, R. J., Kerr, A. D. & Langhorne, P. J. 1996 *Moving Loads on Ice Plates*. Kluwer.

Squire, V. A., Robinson, W. H., Haskell, T. G. & Moore, S. C. 1985 Dynamic strain response of lake and sea ice to moving loads. *Cold Regions Sci. Technol.* **11**, 123–139.

Takizawa, T. 1985 Deflection of a floating sea ice sheet induced by a moving load. *Cold Regions Sci. Technol.* **11**, 171–180.

Takizawa, T. 1988 Response of a floating sea ice sheet to a steadily moving load. *J. Geophys. Res.* **93**, 5100–5112.